Riemann Zeta Function: Recreated in LaTeX from Wikipedia Article

Jake Warde

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Abstract

I have recreated the article mentioned above to learn more about composing mathematics articles using LaTeX. And what a fascinating subject! I certainly don't understand much of it but admire minds that can. Here is the original article:`https://en.wikipedia.org/wiki/Riemann_zeta_function`

1 Introduction

The Riemann zeta function or Euler-Riemann zeta function, $\zeta(s)$, is a function of a complex variable s that analytically continues the sum of the infinite series

$$\zeta(s) - \sum_{i=1}^{\infty} \frac{1}{n^s} \tag{1}$$

which converges when the real part of s is greater than 1. More general representations of $\zeta(s)$ for all s are given below. The Riemann zeta function plays a pivotal role in analytic number theory and has applications in physics, probability theory, and applied statistics.

This function, as a function of a real argument, was introduced and studied by Leonhard Euler in the first half of the eighteenth century without using complex analysis, which was not available at that time. Bernhard Riemann in his article "On the Number of Primes Less Than a Given Magnitude" published in 1859 extended the Euler definition to a complex variable, proved its meromorphic continuation and functional equation and established a relation between its zeros and the distribution of prime numbers.

The values of the Riemann zeta function at even positive integers were computed by Euler. The first of them, $\zeta(s)$ provides a solution to the Basel problem. In 1979 Apéry proved the irrationality of $\zeta(s)$. The values at negative integer points, also found by Euler, are rational numbers and play an important role in the theory of modular forms. Many generalizations of the Riemann zeta function, such as Dirichlet series, Dirichlet L-functions and L-functions, are known.

$$\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \qquad \sigma = \Re(s) > 1. \tag{2}$$

It can also be defined by the integral

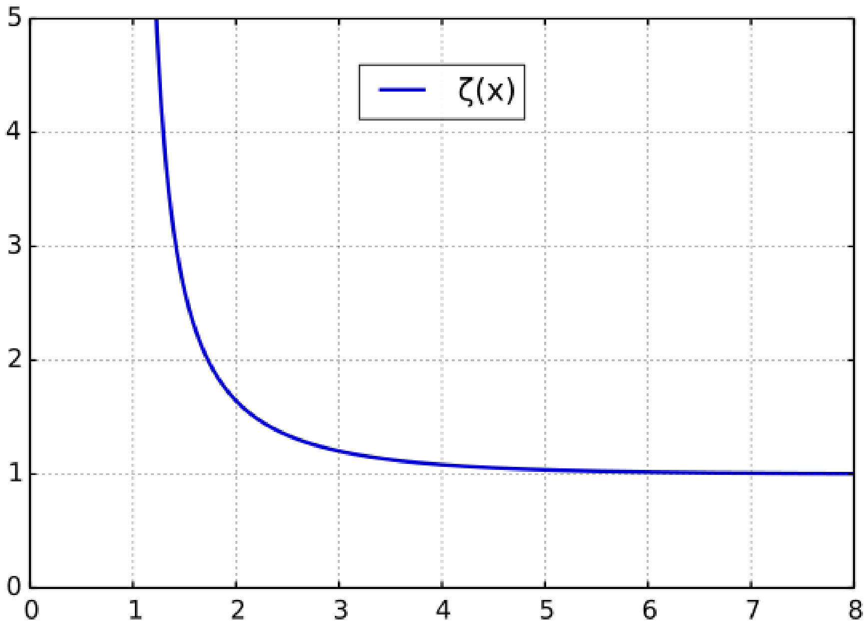
$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^s - 1}{e^x - 1}\, dx \tag{3}$$

where $\Gamma(s)$ is the gamma function. The Riemann zeta function is defined as the analytic continuation of the function defined for $\sigma > 1$ by the sum of the preceding series. Leonhard Euler considered the above series in 1740 for positive integer values of s, and later Chebyshev extended the definition to real $s > 1$. The above series is a prototypical Dirichlet series that converges absolutely to an analytic function for s such that $\sigma > 1$ and diverges for all other values of s. Riemann showed that the function defined by the series on the half-plane of convergence can be continued analytically to all complex values $s/ne1$. For $s = 1$ the series is the harmonic series which diverges to $+^{\infty}$, and

$$\lim_{x\to 1}(s-1)\,(\zeta(s)) = 1 \tag{4}$$

Thus the Riemann zeta function is a meromorphic function on the whole complex s-plane, which is holomorphic everywhere except for a simple pole at $s = 1$ with residue 1.

Figure 1: Riemann zeta function for real s > 1



For any positive even integer 2n:

$$\zeta(2n) = \frac{-n^{n+1}\, B_{2n}(2\pi)^{2n}}{2(2n)!} \tag{5}$$

where $B2n$ is a Bernoulli number. For negative integers, one has

$$\zeta(-n) = -\frac{B_{n-1}}{n+1} \tag{6}$$

for $n \geq 1$, so in particular $\zeta \mathrm{B}_m = 0$ for all odd m other than 1. For odd positive integers, no such simple expression is known, although these values are thought to be related to the algebraic K-theory of the integers; see Special values of L-functions.

Via analytic continuation, one can show that

$$\zeta(-1) = -\frac{1}{12} \tag{7}$$

gives a way to assign a finite result to the divergent series $1+2+3+4+\ldots$ which can be useful in certain contexts such as string theory.

$$\zeta(0) = -\frac{1}{2} \tag{8}$$

$$\zeta(\frac{1}{2}) \approx -1.4603545) \tag{9}$$

This is employed in calculating of kinetic boundary layer problems of linear kinetic equations.

$$\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots = \infty \tag{10}$$

if we approach from numbers larger than 1. Then this is the harmonic series. But its Cauchy principal value

$$\lim_{x\to 0} \frac{\zeta(1+\varepsilon)+\zeta(1-\varepsilon)}{2} \tag{11}$$

exists which is the Euler–Mascheroni constant $\gamma = .05772....$

This ends the sample document. Go to `https://en.wikipedia.org/wiki/Riemann_zeta_function` **to read the entire Wikipedia document.**